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High-gain robust adaptive controllers for multivariable systems

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Abstract: This paper extends the switching free high-gain stabilizing adaptive control rules of Byrnes and Willems to a wide class of adaptive schemes capable of tolerating nonlinear state feedback perturbations.

Keywords: High gain feedback, Adaptive controllers, Robustness, Nonlinear perturbations.

1. Introduction

In a recent paper Byrnes and Willems [2] considered the adaptive stabilization of the m-input/m-output linear time-invariant system in $\mathbb{R}^n$

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \qquad y(t) = Cx(t), \tag{1.1}$$

under the assumption that the system is *minimum phase* with a stable instantaneous gain

$$\sigma(CB) \subset \mathbb{C}^- \tag{1.2}$$

but unknown matrices A, B, C. More precisely it has been shown that the choice of the output-feedback control law

$$u(t) = -k(t)y(t) \tag{1.3}$$

where the time-varying gain $k(t)$ is the monotonically non-decreasing solution of the differential equation

$$\dot{k}(t) = \| y(t) \|^2, \quad k(0) = k_0, \tag{1.4}$$

generates an asymptotically stable closed-loop solution $x(\cdot) \in L_2^n[0, \infty)$ for any state initial condition $x_0 \in \mathbb{R}^n$ and initial gain $k_0 \in \mathbb{R}$. In particular, the gain $k(\cdot)$ is bounded and convergent in the sense that

$$k_\infty = \lim_{t \to \infty} k(t) < +\infty. \tag{1.5}$$

The above mentioned paper [2] and a recently published paper of Mårtensson [5] contain extensions of these results to situations where CB is known to be nonsingular, but not necessarily $\sigma(BC) \subset \mathbb{C}^-$. Then the control law (1.4) has to be modified by the inclusion of 'switching functions'. Overall, these results

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provide an indication of the existence of stabilizing adaptive control schemes of simple form being parametrized by a single gain parameter.

In this paper, we consider extensions of the above controllers for high-gain adaptive stabilization in the *absence of switching* (switching functions play an essential role in the problem of 'learning' the sign (spectrum) of the instanteneous gain, cf. [2,4–7,9,11].

Two questions arise naturally in the context of the adaptation (1.4):

(i) Are there other causal stabilizing gain rules and how can they be constructed?

(ii) Are the results dependent on the linearity of the system resp. the feedback law (1.3) and, if not, what kind of nonlinearities can be tolerated?

For question (ii) consider also Byrnes and Isidori [1]. It is the purpose of this paper to pursue these ideas in more detail. In Section 2 we present technical material describing the dynamics of (1.1) subject to control law (1.3) under the assumption that $k(\cdot)$ is any monotonically non-decreasing function with $k_\infty = \infty$, cf. also [5]. The analysis of 'gain divergence' is used in Section 3 to produce a class of gain adaption rules that ensure gain convergence in the sense of (1.5), together with examples to illustrate how, within this class, it is possible to manipulate the form of the closed-loop responses $x(\cdot)$, $y(\cdot)$ in an arbitrary manner. Finally, in Section 4, the problem of robustness with respect to nonlinearities (disturbances) is considered for the system

$$\dot{x}(t) = Ax(t) + B(u(t) + g(x(t))) \tag{1.6}$$

where $g(\cdot)$ is a nonlinearity of finite gain.

The results extend considerably the results of Byrnes and Willems [2] and Mårtensson [5], indicating the wide choice of adaptive mechanisms available for stabilization and the inherent robustness of the algorithms to nonlinear perturbations.

2. Stability for systems with unbounded gain variation

In this section we present sufficient conditions on A, $D \in \mathbb{R}^{n\times n}$ to ensure exponential stability of the time-varying linear system

$$\dot{x}(t) = (A - k(t)D)x(t), \quad t \geqslant 0, \tag{2.1}$$

for every $k(\cdot) \in \mathscr{K}$, where $\mathscr{K}$ denotes the set of all piecewise continuous functions $k: \mathbb{R}_+ \to \mathbb{R}_+$, monotonically non-decreasing with $\lim_{t\to\infty} k(t) = \infty$.

A time-varying system $\dot{x}(t) = A(t)x(t)$, $t \geqslant 0$, is called *exponentially stable* if its transition matrix $\phi(\cdot, \cdot)$ satisfies for some M, $\omega > 0$,

$$\| \phi(t, t_0) \| \leqslant M\, \mathrm{e}^{-\omega(t-t_0)} \quad \text{for all } t \geqslant t_0 \geqslant 0,$$

and *arbitrarily fast exponentially stable* if for some $\omega(\cdot) \in \mathscr{K}$ we have

$$\| \phi(t, t_0) \| \leqslant M\, \mathrm{e}^{-\omega(t_0)(t-t_0)} \quad \text{for all } t \geqslant t_0 \geqslant 0.$$

Throughout this section the eigenvalues of $A - k(t)D$ are denoted by $\lambda_i(k(t))$, $i = 1, 2, \ldots, n$.

The following proposition shows the dominating role of the eigenvalues of D for arbitrarily fast exponential stability of (2.1).

2.1. Proposition. *For every $k(\cdot) \in \mathscr{K}$ the following conditions are equivalent*:

(i) $\sigma(D) \subset \mathbb{C}_+^\circ = \{s \in \mathbb{C};\ \mathrm{Re}\ s > 0\}$,

(ii) $\lim_{t\to\infty} \mathrm{Re}\ \lambda_i(k(t)) = -\infty$ *for* $i = 1, 2, \ldots, n$;

(iii) *the system* $\dot{x}(t) = (A - k(t)D)x(t)$, $t \geqslant 0$, *is arbitrarily fast exponentially stable.*

Moreover, if one of the conditions (i)–(iii) *is satisfied then there exists* $M' > 0$ *such that for all* $k(\cdot) \in \mathscr{K}$ *there is an* $\omega(\cdot) \in \mathscr{K}$ *and* $t_0^*(k(\cdot)) \geqslant 0$ *with*

$$\| \phi(t, t_0) \|\, M'\, \mathrm{e}^{-\omega(t_0)(t-t_0)} \quad \textit{for all } t \geqslant t_0 \geqslant t_0^*. \tag{2.2}$$

Proof. (i) $\Rightarrow$ (ii): Select $k_1 > 0$ and $T \in \mathrm{GL}(n, \mathbb{C})$ such that

$$T(k_1 D)T^{-1} = \begin{bmatrix} \lambda & \delta_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \delta_{n-1} \\ 0 & & & \lambda_n \end{bmatrix}$$

and Re $\lambda_i > 1$, $\delta_i \in \{0, 1\}$ for $i = 1, 2, \ldots, n$. The spectrum of $A - k(t)D$ and

$$\hat{A} + \hat{D}(t) := TAT^{-1} + \left(-\frac{k(t)}{k_1}\right)Tk_1DT^{-1}$$

coincide and by Gerschgorin's Circle Theorem (see e.g. [8]) we have

$$\sigma(\hat{A} + \hat{D}(t)) \subset \bigcup_{i=1}^{n} \mathscr{C}_i(t)$$

where

$$\mathscr{C}_i(t) := \left\{ \mu(t) \in \mathbb{C} : \left| -\frac{k(t)}{k_1}\lambda_i + \hat{a}_{ii} - \mu(t) \right| \leqslant \sum_{\substack{j=1 \\ j \neq 1}}^{n} |\hat{a}_{ij}| + \frac{k(t)}{k_1}\delta_i \right\}.$$

Since Re $\lambda_i > 1$ we conclude that $\lim_{t \to \infty}$ Re $\mu(t) = -\infty$ for every $\mu(t) \in \mathscr{C}_i(t)$, whence (ii) follows.

(ii) $\Rightarrow$ (i): Assume there is an eigenvalue $\tilde{\lambda}_i$ of D with Re $\tilde{\lambda}_i \leqslant 0$. Then the real part of the corresponding eigenvalue $k(t)\tilde{\lambda}_i$ of $k(t)D$ either remains 0 or tends to $-\infty$ as $t \to \infty$. Thus by Gerschgorin's Theorem there exists $t^* > 0$ such that at least for one $\lambda_i(k(t))$ we have Re $\lambda_i(k(t) \geqslant N > -\infty$ for all $t > t^*$ which contradicts (ii).

(i) $\Rightarrow$ (iii): Let $p = P^{\mathrm{T}} > 0$ and $Q = Q^{\mathrm{T}} > 0$ such that $D^{\mathrm{T}}P + PD = \mathrm{Q}$ and let $t' \in \mathbb{R}$ such that $A(t) := A - k(t)D$ is exponentially stable for every $t \geqslant t'$. We prove that $V(x(t)) := x^{\mathrm{T}}(t)Px(t)$ is a Liapunov function for (2.1). Set $A' = A - k(t')D$; then differentiation yields

$$\dot{V}(x(t)) = x^{\mathrm{T}}(t)[A'^{\mathrm{T}}P + PA']x(t) - (k(t) - k(t'))x^{\mathrm{T}}(t)Qx(t).$$

Since there exist q_1, p_2, $q_3 > 0$ such that $q_1 \|x\|^2 \leqslant x^{\mathrm{T}}Qx$, $x^{\mathrm{T}}(A'^{\mathrm{T}}P + PA')x \leqslant p_2 \|x\|^2$, $q_3 \|x\|^2 \leqslant x^{\mathrm{T}}Px$ we obtain

$$\dot{V}(x(t)) \leqslant (p_2 - (k(t) - k(t'))q_1) \|x\|^2 \leqslant -\omega(t)V(x(t)) \quad \text{for } t \geqslant t_0^*(k(\cdot)) \tag{2.3}$$

where t_0^* is taken such that

$$\omega(t) := = \frac{1}{q_3}(p_2 - (k(t) - k(t'))q_1) > 0 \quad \text{for all } t \geqslant t_0^*.$$

This proves (iii).

(iii) $\Rightarrow$ (i): It suffices to prove (i) for the case where $A(\cdot) \equiv 0$, because $\lim_{t_0 \to \infty} \omega(t_0) = \infty$ allows to apply a disturbance result of Coppel [3, Proposition 1]. However if some eigenvalue $\tilde{\lambda}$ of D satisfies Re $\tilde{\lambda} \leqslant 0$ then $\dot{x}(t) = -k(t)Dx(t)$ is unstable.

In order to show the last claim of Proposition 2.1 we conclude from (2.4),

$$V(x(t)) \leqslant \mathrm{e}^{-\omega(t_0)(t-t_0)}V(x(t_0)) \quad \text{for all } t \geqslant t_0 \geqslant t_0^*$$

and because of (2.3c),

$$\|x(t)\| \leqslant \frac{\sqrt{p_3}}{\sqrt{q_3}}\mathrm{e}^{-\omega(t_0)(t-t_0)/2}\|x(t_0)\| \quad \text{for all } t \geqslant t_0 \geqslant t_0^*. \qquad \square$$

Now we are in a position to prove the main result of this section. Theorem 2.2 is the essential tool to produce a fairly rich class of stabilizing gain adaptation rules which will be done in the following section. The theorem itself is contained in the (recently published) thesis of Mårtensson [6]. We discovered it independently and give a completely different proof. Recall that an eigenvalue of a matrix is called *semisimple* if all corresponding blocks in the Jordan canonical form are of size 1×1.

2.2. Theorem (Unbounded gain variation theorem). *Consider the time-varying system*

$$\dot{x}(t) = (A - k(t)D)x(t), \quad t \geq 0, \tag{2.4}$$

and assume the following conditions are satisfied:

(i) *There exist* k^*, $\varepsilon > 0$ *such that* $\sigma(A - k(t)D) \subset \{s \in \mathbb{C} \mid \operatorname{Re} s < -\varepsilon\}$ *for all* $k(t) \geq k^*$.

(ii) *If* $0 \in \sigma(D)$ *then* 0 *is semisimple.*

Then (2.6) *is exponentially stable for every* $k(\cdot) \in \mathscr{K}$. *Moreover, there exists* $M' > 0$ *such that for every* $\varepsilon > 0$ *and every* $k(\cdot) \in \mathscr{K}$ *there is* $t_0^*(\varepsilon, k(\cdot))$ *such that the transition map* $\phi(\cdot, \cdot)$ *of* (2.6) *satisfies*

$$\| \phi(t, t_0) \| \leq M' \, \mathrm{e}^{-\varepsilon(t-t_0)/2} \quad \forall t \geq t_0 \geq t_0^*. \tag{2.5}$$

Proof. The invariance of exponential stability with respect to constant coordinate transformations together with (ii) implies that we can assume D to be of the form

$$D = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where } \Lambda = \begin{bmatrix} \lambda_1 & \delta_1 & & \\ & \ddots & \ddots & \\ & & \ddots & \delta_{l-1} \\ & & & \lambda_l \end{bmatrix}, \quad \lambda_i \neq 0, \ \delta_i \in \{0, 1\}.$$

Then

$$A - k(t)D = \begin{bmatrix} A_1 - k(t)\Lambda & A_2 \\ A_3 & A_4 \end{bmatrix} \tag{2.6}$$

and we choose $S \in \mathrm{Gl}(n-l, \mathbb{C})$ such that

$$A_4^J := SA_4S^{-1} = \begin{bmatrix} \lambda_{l+1} & \delta_{l+1} & & \\ & \ddots & \ddots & \\ & & \ddots & \delta_{n-1} \\ & & & \lambda_n \end{bmatrix}, \quad \delta_j \in \{0, 1\}.$$

For $r > 0$ and $T = \mathrm{diag}(\alpha_{l+1}, \ldots, \alpha_n) \in \mathrm{Gl}(n-l, \mathbb{R})$, we obtain

$$\begin{bmatrix} I_s & 0 \\ 0 & rST \end{bmatrix} [A - k(t)D] \begin{bmatrix} I_s & 0 \\ 0 & (rST)^{-1} \end{bmatrix} = \begin{bmatrix} A_1 - k(t)\Lambda & A_2(rST)^{-1} \\ rSTA_3 & TA_4^JT^{-1} \end{bmatrix}$$

where

$$TA_4^JT^{-1} = \begin{bmatrix} \lambda_{l+1} & \frac{a_{l+1}}{a_{l+2}}\delta_{l+1} & & \\ & \ddots & \ddots & \\ & & \ddots & \frac{a_{n-1}}{a_n}\delta_{n-1} \\ & & & \lambda_n \end{bmatrix}. \tag{2.7}$$

For r sufficiently small and α_i suitable chosen, Gerschgorin's Theorem together with (i) implies

$$\operatorname{Re} \lambda_i(k(t)) \to -\infty \quad \text{as } t \to \infty, \quad i = 1, 2, \dots, l,$$

and

$$\operatorname{Re} \lambda_i < -\tfrac{3}{4}\varepsilon \quad \text{for } i = l+1, \dots, n,$$

for every $k(\cdot) \in \mathscr{K}$. Now by Variation-of-Constants it can be shown that (2.4) is exponentially stable and (2.5) holds true. □

2.3. Remark. It is well known that CB invertible implies condition (ii) of Theorem 2.2. Furthermore condition (i) yields that the system (A, B, C) is 'minimum phase' (cf. [10]).

3. A general class of adaptive controllers

In this section we consider a class of systems $\mathscr{S}$ described by the matrix differential equation in $\mathbb{R}^{n \times l}$

$$\dot{X}(t) = AX(t) + BU(t), \quad X(0) = X_0 \in \mathbb{R}^{n \times l}, \tag{3.1a}$$

with output equation

$$Y(t) = CX(t) \tag{3.1b}$$

and measurement equation

$$Z(t) = FX(t) \tag{3.1c}$$

where $X(t) \in \mathbb{R}^{n \times l}$, $U(t) \in \mathbb{R}^{m \times l}$, $Y(t) \in \mathbb{R}^{p \times l}$ and $Z(t) \in \mathbb{R}^{q \times l}$ and A, B, C, F are constant matrices of appropriate sizes. Furthermore we assume:

$$\operatorname{rank} F = q, \tag{3.2}$$

$$\det CB \neq 0, \tag{3.3}$$

$$\exists k^* > 0,\ \varepsilon > 0: \quad \operatorname{Re} \lambda_i(k) < -\varepsilon \quad \text{for all } k \geq k^*, \quad i = 1, 2, \dots, n, \tag{3.4}$$

where $\lambda_i(k)$ denote the eigenvalues of $A - kBC$, $i = 1, 2, \dots, n$.

For $\mathscr{S}$ we analyze a wide class $\hat{\mathscr{K}}$ of time-varying high-gain output feedback controllers

$$U(t) = -k(t)Y(t) \tag{3.5}$$

parametrized by a scalar time-varying gain $k(\cdot)$ where the closed-loop system

$$\dot{X}(t) = (A - k(t)BC)X(t), \quad X(0) = X_0 \in \mathbb{R}^{n \times l}, \tag{3.6}$$

has desired stability properties and the gain converges in the sense that

$$\lim_{t \to \infty} k(t) = k_\infty < \infty. \tag{3.7}$$

As an example let for instance $F = C$ and

$$k(t) = k(0) + \int_0^t \| Y(s) \|^p \, ds. \tag{3.8}$$

Then Byrnes and Willems [2] have shown for $p = 2$ and $l = 1$ that (3.7) holds and $y(\cdot) \in L_p^m(0, \infty)$. In the following it will be shown that not only this result holds for arbitrary p but holds for a wider class of adaptive mechanisms as well. The construction of this class is one of the main contribution of this paper.

We will need to define linear vector spaces for technical reasons:

$$\mathscr{C}^{q\times l} := \{ Z(\cdot) : \mathbb{R} \to \mathbb{R}^{q\times l} \mid Z(\cdot) \text{ is a continuous function} \},$$

$$\mathscr{E}^{q\times l} := \{ Z(\cdot) \in \mathscr{C}^{q\times l} \mid \| Z(t) \| \leqslant M \, \mathrm{e}^{-\alpha t} \text{ for all } t \geqslant 0 \text{ and some } \alpha > 0,\ M > 0 \}.$$

We also define $\mathscr{R}$ to be a specified linear vector space of acceptable responses of the measurement vector Z with the property that

$$\mathscr{E}^{q\times l} \subset \mathscr{R} \subset \mathscr{C}^{q\times l}.$$

The choice of $\mathscr{R}$ is open to the designer to produce the desirable responses. We will concentrate on control laws of the form (3.5) where $k(\cdot)$ is an element of the set $\hat{\mathscr{K}}(\mathscr{R})$ of stabilizing gains

$$\psi : \mathscr{C}^{q\times l} \to \mathscr{C}, \quad Z(\cdot) \mapsto k(\cdot), \tag{3.9}$$

which satisfy the following conditions:

$$\psi(\mathscr{R}) \subset L_\infty[0, \infty), \tag{3.10}$$

$$\psi(Z) \in L_\infty[0, \infty) \ \Rightarrow\ Z \in \mathscr{R}, \tag{3.11}$$

$$\psi(Z)(t) \text{ is monotonically non-decreasing for all } t \geqslant 0 \text{ and every } Z \in \mathscr{C}^{q\times l}. \tag{3.12}$$

We note that the map $\psi : Y(\cdot) \mapsto k(\cdot)$ defined by (3.8) belongs to $\hat{\mathscr{K}}(L_p^{m\times l}(0, \infty))$, so that the Byrnes–Willems controller is just one element of $\hat{\mathscr{K}}(L_2^m(0, \infty))$. The following theorem is the main theorem of this paper and demonstrates that *any* element of $\hat{\mathscr{K}}(\mathscr{R})$ is capable of ensuring gain convergence in the sense of (3.7) with measurement responses $Z(\cdot) \in \mathscr{R}$. Examples are given later in this section.

3.1. Theorem (High-gain adaptive stabilization). *Let $\psi(\cdot)$ be any element of $\hat{\mathscr{K}}(\mathscr{R})$. Then given any system* (3.1) *with control* (3.5) *and gain adaptation $k(\cdot) = \psi(Z(\cdot))$ the response of the closed-loop system* (3.6) *has the property that $Z(\cdot) \in \mathscr{R}$ and moreover the gain $k(t)$ satisfies* (3.7).

Proof. Assume $\lim_{t\to\infty} k(t) = k_\infty < \infty$. Then $\psi(Z(\cdot)) \in L_\infty[0, \infty)$ and by (3.11) $Z(\cdot) \in \mathscr{R}$. Assume conversely that $k(t)$ diverges. Then by Theorem 3.4, $X(\cdot) \in \mathscr{E}^{q\times l}$ and $Z(\cdot) = FX(\cdot) \in \mathscr{E}^{q\times l} \subset \mathscr{R}$ and hence by (3.10), $\psi(Z(\cdot) \in L_\infty[0, \infty)$ which implies that k_∞ exists and is finite by monotonicity (contradiction). □

In order to illustrate the wide range of adaptive mechanisms available from the above theorem, we consider several examples:

3.2. Examples. (a) Taking $\mathscr{R} = L_p^{m\times l}[0, \infty)$ and $Z = Y$ the mechanism (3.8) ensures that the gain $k(\cdot)$ converges and $Y(\cdot) \in L_p^{m\times l}(0, \infty)$ (independent of the choice of $p \geqslant 1$).

(b) Let $h : \mathbb{R} \to \mathbb{R}$ be a non-zero polynomial

$$h(\lambda) = h_1\lambda^1 + h_2\lambda^2 + \cdots + h_n\lambda^n, \quad h_i \geqslant 0. \tag{3.13}$$

Define

$$\mathscr{R} := \bigcap_{\{i, h_i \neq 0\}} L_i^{q\times l}(0, \infty). \tag{3.14}$$

Then the map $\psi : Z(\cdot) \mapsto k(\cdot)$ defined by

$$\dot{k}(t) = h(\| Z(t) \|), \quad k(0) = k_0, \tag{3.15}$$

is in $\hat{\mathscr{K}}(\mathscr{R})$. The theorem here applies to produce a convergent gain with measurement response $Z(\cdot) \in \mathscr{R}$.

(c) Let $\psi: Z(\cdot) \mapsto k(\cdot)$ be defined by

$$k(t) = k_0 + \alpha \max_{0 \leqslant s \leqslant t} \| Z(s) \|, \quad \alpha > 0, \tag{3.16}$$

then $\psi(\cdot) \in \hat{\mathscr{K}}(L_\infty^{q \times l}[0, \infty))$. The induced adaptive scheme is hence stable with $Z(\cdot) \in L_\infty^{q \times l}[0, \infty) = \mathscr{R}$.

(d) With $\psi: Z(\cdot) \mapsto k(\cdot)$ defined by

$$\dot{\xi}(t) = h(\| Z(t) \|), \quad \xi(0) = \xi_0, \tag{3.17a}$$

$$k(t) = \xi(t) + \alpha \max_{0 \leqslant s \leqslant t} \| Z(s) \|, \quad \alpha \geqslant 0, \tag{3.17b}$$

where $h(\cdot)$ is defined as in (b), $\psi(\cdot) \in \hat{\mathscr{K}}(\mathscr{R})$ where

$$\mathscr{R} = L_\infty^{q \times l}[0, \infty) \cap \bigcap_{i=1}^{n} L_i^{q \times l}[0, \infty)$$

and the adaptive scheme converges with $Z(\cdot) \in \mathscr{R}$.

This example contains the others as special cases.

4. Robustness of the adaptive controller with respect to nonlinear perturbations

Robustness is an important question of current concern to the control community. In this section we consider linear systems in $\mathscr{S}$ perturbed by a form of nonlinear state feedback

$$U(t) = -k(t)Y(t) + (g(X))(t) \tag{4.1}$$

where $g(\cdot): \mathscr{R} \to \mathscr{R}$ is a nonlinear causal map with the property

$$\| g(X(\cdot) \|_{\mathscr{R}} \leqslant g_0 \| X(\cdot) \|_{\mathscr{R}} \quad \forall X(\cdot) \in \mathscr{R} \tag{4.2}$$

where $\| \cdot \|_{\mathscr{R}}$ denotes any norm on $\mathscr{R}$ and $g_0 \geqslant 0$.

4.1. Theorem (Robustness). *Let $\mathscr{R}$ be any finite intersection of $L_p(0, \infty)$ spaces and $\psi(\cdot) \in \hat{\mathscr{K}}(\mathscr{R})$. Then given any systems in $\mathscr{S}$ with control of the form* (4.1) *and $k(\cdot) = \psi(z(\cdot))$ there exists $\alpha > 0$ (independent of ψ and X_0) such that for every nonlinearity of the class satisfying*

$$\alpha g_0 < 1 \tag{4.3}$$

the response of the nonlinear closed-loop system

$$\dot{X}(t) = (A - k(t)BC)X(t) + B(g(X)(t), \quad X(0) = X_0, \qquad z(t) = FX(t), \tag{4.4}$$

has the properties

$$z(\cdot) \in \mathscr{R}, \tag{4.5}$$

$$\lim_{t \to \infty} k(t) = k_\infty < \infty. \tag{4.6}$$

Proof. If $\lim_{t \to \infty} k(t) = k_\infty < 0$ then $\psi(z(\cdot)) \in L_\infty[0, \infty)$ and because $\psi(\cdot) \in \hat{\mathscr{K}}(\mathscr{R})$, (3.11) implies $Z(\cdot) \in \mathscr{R}$. If otherwise $k(t) \to +\infty$ for $t \to \infty$ then there exist M, $t_0^*(k(\cdot)) > 0$ such that

$$\| \phi(t, t_0) \| \leqslant M \,\mathrm{e}^{-\lambda(t - t_0)} \quad \text{for } t \geqslant t_0 \geqslant t_0^*(k(\cdot)) \tag{4.7}$$

where $\phi(t, t_0)$ is the transition matrix of

$$\dot{X}(t) = (A - k(t)BC)X(t).$$

Applying Variation-of-Constants to (4.4) yields

$$X(t)=\phi(t, t_0)X(t_0)+L_{t_0}g(x)(t).$$

where

$$L_{t_0}: V(\cdot) \mapsto \int_{t_0}^{\cdot} \phi(\cdot, s)BV(s)\, \mathrm{d}s.$$

Then, for $t_0 \geqslant t_0^*(k)$,

$$\| X(\cdot) \|_{\mathscr{R}} \leqslant \| \phi(\cdot, t_0)X(t_0) \|_{\mathscr{R}} + \| L_{t_0} \|_{\mathscr{R}} g_0 \| X(\cdot) \|_{\mathscr{R}}.$$

Clearly because of (4.6) there exists $\alpha > 0$, $\beta > 0$ such that

$$\| L_{t_0} \| \leqslant \alpha \quad \text{and} \quad \| \phi(\cdot, t_0)X(t_0) \| \leqslant \beta \;\; \forall t_0 \geqslant t_0^*(k).$$

Since $\alpha g_0 < 1$ we obtain

$$\| X(\cdot) \| \leqslant \frac{\beta}{1-\alpha g_0} < \infty,$$

i.e. $X(\cdot) \in \mathscr{R}$ and $z(\cdot) = FX(\cdot) \in \mathscr{R}$. However then $k(\cdot) = \psi(z(\cdot)) \in L_\infty[0, \infty)$ and so k_∞ exists, which is a contradiction.

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